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COMMENT

Generalised depinning transition in a solid-on-solid model†

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Abstract. The pinning of a domain wall by a row of weakened bonds in the *interior* of a planar Ising ferromagnet is studied in the solid-on-solid limit. Analysing a continuum version of the model we show that a pinning–depinning transition occurs whenever the couplings on the two sides of the defect are different. The roughening transition found by Abraham is contained as a special case.

Critical behaviour near defect planes and free surfaces in an otherwise homogeneous system has been the subject of much recent scrutiny. In particular, in two-dimensional Ising models, such defects have been studied in the context of surface and interface phenomena (Fisher and Ferdinand 1967, Watson 1972, Fisher and de Gennes 1978, Burkhardt and Eisenriegler 1981), pinning–depinning transitions (Abraham 1980, 1981a, b, Burkhardt 1981, Chalker 1981, Kroll 1981, Chui and Weeks 1981, van Leeuwen and Hilhorst 1981) (in $D=3$ this was studied by Pandit *et al* (1982); see also Nakanishi and Fisher (1982)), and non-universal behaviour associated with an internal defect line (Bariev 1979, McCoy and Perk 1980). The purpose of this comment is to propose a more general type of depinning transition and calculate the corresponding phase diagram. Specifically, if the defect is viewed as a mediator between two differently ordered physical systems, then the depinning transition is a transition in which one system begins to promote its own order into the other system. This we regard as a generalisation of the depinning transition concept which, to our knowledge, has not been studied before.

In order to gain further orientation and to make our statement of purpose more precise it is helpful to first define a model. Consider a two-dimensional, square, ferromagnetic, nearest-neighbour Ising model with a 'seam' of defect couplings, as shown schematically in figure 1. On one side of the defect the couplings have values $K_1 = J_1/k_B T$, and on the other, values $K_2 = J_2/k_B T$. The defect couplings have values $K_d = J_d/k_B T$. Generally, we will assume that $K_1 \neq K_2$ and in this respect our model differs from those usually studied (Fisher and Ferdinand 1967, Watson 1972, Burkhardt and Eisenriegler 1981, Abraham 1980, 1981a, b; for a recent review see Fisher 1984). Our model is most properly thought of as a model of two thermodynamic systems, K_1 and K_2, interacting via K_d. However, the connection with the models previously studied can be made by taking antiperiodic boundary conditions and, for example,

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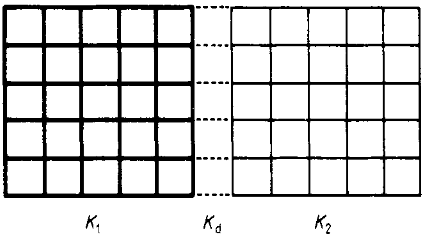

Figure 1. Couplings in the two Ising models interacting via defect couplings: thick lines, K_1; thin lines, K_2; dotted lines, K_d.

$K_1 = \infty$, $K_2 < \infty$, $K_d < K_2$. In this case we have a 'seam' of (weakened) defect couplings K_d next to the fully ordered system K_1 which acts as a 'hard surface'. By setting $K_d = aK_2$, where a is some parameter $0 < a < 1$, we get the model precisely analogous to that studied by Abraham (1980, 1981a, b) in the context of the depinning transition. For this model it has been exactly shown by Abraham (1980, 1981a, b) that the interface, created by the boundary conditions, remains pinned to the defect at sufficiently low temperatures but, as the temperature is increased above a certain value $T_R(a)$, the interface depins and, in the thermodynamic limit, starts to wander infinitely far away from the defect. Exactly at the depinning transition temperature $T_R(a)$ the defect specific heat exhibits a jump discontinuity induced by the extra degrees of freedom available for the depinning interface. However, if the defect is internal (that is, not located at the boundary) then the interface will remain pinned to the defect at all temperatures below the critical temperature T_c (Abraham 1980, 1981a, b). Note also that the singularity in the *surface* specific heat (Fisher and Ferdinand 1967, Watson 1972) at T_c is the special case of the depinning transition at $T_R(a = 0) = T_c$.

Now let us consider the general case $K_1 \neq K_2$ when both K_1 and K_2 are finite (without loss of generality we shall take $K_2 < K_1$ throughout this work). Furthermore, let $K_d = aK_2$, with parameter a as defined above, and let us consider a situation when $K_2 > K_c$ (i.e., both systems K_1 and K_2 are ordered). One can think of the system K_1 as being 'more ordered' than the system K_2. The principal result of this work is the following: at sufficiently low temperatures the interface between the two regions will be pinned to the defect. However, there is a temperature $T_R(a)$ at which the interface depins into the less ordered region K_2, in a manner precisely analogous to the depinning transition studied by Abraham (1980, 1981a, b). Note that, even though the defect is internal in our model, the depinning transition will still take place *provided* $K_1 \neq K_2$. The depinning transition temperature $T_R(a)$ will also depend on the ratio $K_1/K_2 = \alpha$, $\alpha > 1$, and in what follows we shall use the notation $T_R(a, \alpha)$ to denote this dependence explicitly. At this temperature the 'more ordered' system K_1 starts promoting its own order into the 'less ordered' system K_2. Clearly, when $K_1 = K_2$, the two systems are 'equally ordered' and the depinning transition will not take place, in agreement with the exact result of Abraham (1980, 1981a, b). Simply, in this case the interface cannot 'depin' because of the symmetric situation on two sides of the defect.

Quantitatively, the generalised depinning transition can be studied in several ways, including exact calculations which are quite involved technically (Wolff 1983), renormalisation group methods (Švrakić 1982), or in the solid-on-solid (SOS) limit (Burkhardt 1981, Chalker 1981, Kroll 1981, Chui and Weeks 1981, van Leeuwen and Hilhorst 1981). Of these, the SOS calculation is most transparent since the problem can be reduced to the elementary quantum-mechanical problem of finding the bound states of a particle in a well. In the rest of this comment the SOS calculation of the depinning transition phase diagram $T_R(a, \alpha)$ is illustrated. Our main result is given by equation (12), and is shown in figure 2.

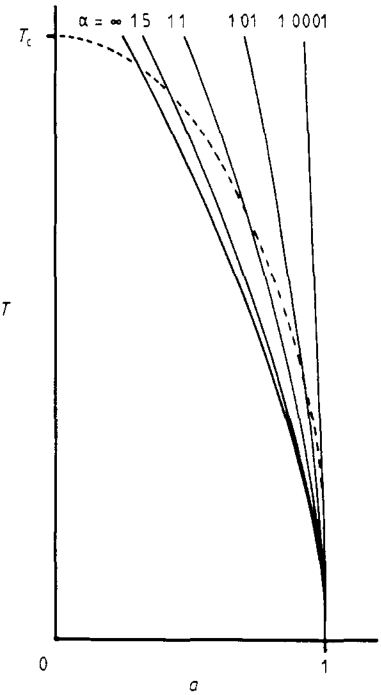

Figure 2. Depinning transition temperature $T_R(a, \alpha)$ (12) shown as a function of a for various values of α. The broken curve shows Abraham's exact result for $\alpha = \infty$.

In the SOS limit, overhangs in the interface are suppressed and the canonical form of the interface energy is, (cf Müller-Hartmann and Zittartz 1977)

$$E(x) = 2K_2\left(\sum_{i=1}^{N-1} |A(x_{i+1})x_{i+1} - A(x_i)x_i| + \sum_{i=1}^{N} B(x_i)\right) \tag{1}$$

where x_i denotes the perpendicular distance of the interface at the ith column from the defect at $x = 0$. The x_i vary continuously in the interval $-\infty < x_i < \infty$. The case of a discrete spectrum for the x_i can also be treated exactly, but will not be discussed here, as the phase transition is qualitatively the same (Wolff 1983). The functions $A(x)$ and $B(x)$ arise from the various couplings in the system; explicitly we have

$$A(x) = \begin{cases} 1 & x \geq 0 \\ \alpha & x < 0 \end{cases} \tag{2}$$

and

$$B(x) = \begin{cases} 1 & x \geq 1 \\ a & 0 \leq x < 1 \\ \alpha & x < 0. \end{cases} \tag{3}$$

The special case of a hard surface (Abraham 1980, 1981a, b, Burkhardt 1981, Chalker 1981, Kroll 1981, Chui and Weeks 1981, van Leeuwen and Hilhorst 1981) corresponds to $\alpha = \infty$ or, equivalently, to the restriction $x_i \geq 0$.

The transfer matrix (Huang 1963) corresponding to (1) is given by

$$T(x, y) = \exp[-2K_2(|A(x)x - A(y)y| + B(y))]. \tag{4}$$

Representing the eigenfunctions and eigenvalues of T as

$$\int_{-\infty}^{\infty} \mathrm{d}y\, T(x, y)\psi(y) = \lambda\psi(x) \tag{5}$$

we find by standard arguments (Huang 1963) that the interface free energy F_I in the limit $N \to \infty$ is given by

$$F_\mathrm{I}/N = -k_\mathrm{B}T \ln \lambda_\mathrm{m} \tag{6}$$

where λ_m is the largest eigenvalue of T with eigenfunction $\psi_\mathrm{m}(x)$. The probability density $P(x)$ for finding the interface at a distance x from the defect is then

$$P(x) \sim \exp(-2K_2B(x))|\psi_\mathrm{m}(x)|^2. \tag{7}$$

Using $(-\mathrm{d}^2/\mathrm{d}x^2 + K^2)\exp(-K|x-y|) = 2K\delta(x-y)$ in (5), we find that $\psi(x)$ satisfies the differential equation

$$(-\mathrm{d}^2/\mathrm{d}x^2 + V(x))\psi(x) = 0 \tag{8}$$

with

$$V(x) = 4K_2A(x)(K_2A(x) - \lambda^{-1}\exp(-2K_2B(x))) \tag{9}$$

and the boundary condition $\psi'(0-) = \alpha\psi'(0+)$. (This boundary condition is equivalent to adding a δ-function constant$\cdot\delta(x)$ to $V(x)$ in (9).) Equations (8) and (9) can be viewed as the one-dimensional Schrödinger equation for a particle moving in the potential $V(x)$.

From elementary quantum mechanics (Messiah 1969) one knows that (8) and (9) have two types of solution for $x > 1$ depending on whether $V(x)$ is positive or negative, namely scattering solutions $\psi_\mathrm{s}(x) \sim \sin(kx + \delta)$ with eigenvalues

$$\lambda_\mathrm{s} = 4K_2\exp(-2K_2)/(4K_2 + k^2) \tag{10}$$

and bound-state solutions $\psi_\mathrm{b}(x) \sim \exp(-\kappa x)$ with eigenvalues

$$\lambda_\mathrm{b} = 4K_2\exp(-2K_2)/(4K_2 - \kappa^2). \tag{11}$$

The resulting spectrum of the transfer operator $T(x, y)$ (5) is shown schematically in figure 3. From the eigenvalue expressions for $\lambda_\mathrm{s}(k)$ (10) and $\lambda_\mathrm{b}(\kappa)$ (11) it is evident that the largest λ for which (8) and (9) have a solution corresponds to the most tightly bound state, i.e. the state with largest κ, and, in the absence of bound states for $x > 1$, to the $k = 0$ scattering state. Because of (7) these two types of solution correspond to pinned and depinned interfaces respectively. At temperatures where the eigenvalues of these two types of solution become degenerate the system exhibits a depinning transition and the interface depins into the less ordered region K_2. The depinning temperature $T_\mathrm{R}(a, \alpha)$ is found to be determined by

$$2K_2(\exp[2(1-a)K_2] - 1)^{1/2} = \tan^{-1}[(1 - \alpha^{-1}\exp[(2(1-\alpha)K_2])(\exp[2(1-a)K_2] - 1)^{-1}]^{1/2} \tag{12}$$

and is shown in figure 2. For $\alpha \to \infty$ equation (12) reduces to the special case studied by Burkhardt (1981) and by Chalker (1981). For $\alpha = 1$ the only solution of (12) is

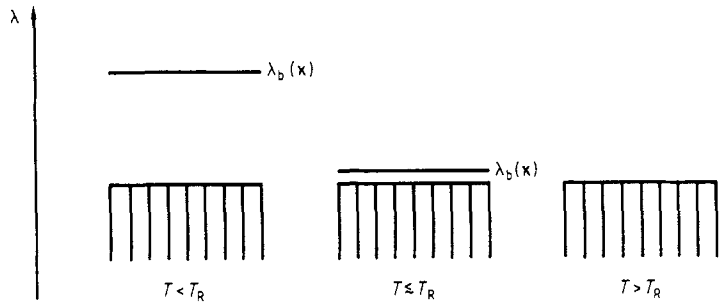

Figure 3. Spectrum of the transfer operator $T(x, y)$ (5). The vertical bars represent the continuum of the scattering state eigenvalues $\lambda_s(k)$ (10); the horizontal lines represent the bound-state eigenvalue $\lambda_b(\kappa)$ (11) (only the largest one is shown).

given by $T_R=\infty$, i.e. the interface remains pinned to the defect at all temperatures in agreement with exact results (Abraham 1980, 1981a, b).

Since the gap between the eigenvalues $\lambda_b(\kappa)$ (11) of the bound state and $\lambda_s(0)$ (10) of the $k=0$ scattering state varies as $(T_R-T)^2$ for $T\to T_R$ the interface specific heat exhibits a jump discontinuity at the transition. One can show from (7) that both the mean distance $\langle x\rangle$ of the interface from the defect and the root-mean square width $\langle(x-\langle x\rangle)^2\rangle$ diverge as $(T_R-T)^{-1}$ for $T\to T_R$ similar to the behaviour found by Abraham (1980, 1981a, b) for the special case $\alpha=\infty$. The height–height correlation function $\langle(x_n-x_0)^2\rangle$ approaches its $n\to\infty$ limit $2\langle(x-\langle x\rangle)^2\rangle$ with an exponential tail $\exp[-\text{constant}\cdot n\cdot(T_R-T)^2]$ for $T\leqslant T_R$ and is proportional to n for $T>T_R$.

In summary, we have investigated the influence of an internal defect on the interface in a two-dimensional Ising model in the SOS limit. It was found that whenever the couplings on the two sides of the defect are different the interface becomes rough and depins for the defect into the less ordered region if the temperature exceeds a critical value $T_R(a, \alpha)$. The results reported above have been obtained by studying a continuum version of the SOS limit which is known to give a qualitatively correct picture of the transition for the special cases $\alpha=1$ and $\alpha=\infty$, respectively (Burkhardt 1981, Chalker 1981, Kroll 1981, Chui and Weeks 1981, van Leeuwen and Hilhost 1981). A quantitative improvement can be obtained by allowing the spectrum of the x_i to be discrete (Wolff 1983).

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